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Degree of Approximation of Differentiable Functions by Reciprocals of Polynomials on [0, ∞)

R. K. BEATSON

University of Texas, Austin, Texas 78712

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Many results are known concerning the degree of approximation of differentiable functions by reciprocals of polynomials on $[0, \infty)$. However most of these results concern approximation of $1/f$ where

$$f^{(j)}(x) \geqslant 0, \qquad x \geqslant 0, \qquad j = 0, 1, 2,... \tag{0.1}$$

(see, for example, [7]). This paper extends these results in two directions: weakening the positivity condition (0.1) for entire functions, and showing some results for functions with only a finite number of derivatives.

1. ORDER OF APPROXIMATION RESULTS FOR CERTAIN ENTIRE FUNCTIONS

As usual we write

$$\lambda_{0,n}(f^{-1}) = \inf_{p \in \pi_n} \left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{[0,\infty)},$$

where throughout $\| \cdot \|_I$ indicates the uniform norm on the interval I, and π_n is the class of polynomials of degree $\leqslant n$.

Some of the known results on the order of approximation of reciprocals $1/f$ of entire functions follow from Taylor series expansion of the function f about zero. The basis of these arguments is: Take p_n^*, the Maclaurin polynomial of degree n; choose upper end points $r(n)$, and discuss

$$\|f - p_n^*\|_{[0,r(n)]}, \qquad \inf_{x \geqslant r(n)} f(x), \ \inf_{x \geqslant r(n)} p_n^*(x)$$

using the positivity conditions to deduce that $p_n^{*\prime}(x) \geqslant 0, x \geqslant 0$. Examination shows that use of the same argument, with a Taylor expansion, p_n, about $r(n)$

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or $r(n)/2$, will allow the positivity conditions to be weakened; given only a satisfactory estimate of $\|f - p_n\|_{[0,r(n)]}$. Thus in Theorems 1.1 and 1.2 the power of n, -1, $-1 - 1/\Lambda$, respectively, is known to be best possible [5, 6].

If $f(x)$ has $n+1$ continuous derivatives on $[0, r]$ then with $p_n(x) = \sum_{j=0}^{n} (f^{(j)}(r)(x-r)^j/j!)$,

$$\|f - p_n\|_{[0,r]} \leqslant \|f^{(n+1)}\|_{[0,r]}\, r^{n+1}/(n+1)!\,, \tag{1.1}$$

a classical formula for the error in truncated Taylor series expansion. If also $f(z)$ is entire then by Cauchy's inequalities (see, e.g., [4, p. 202]).

$$\|f^{(n+1)}\|_{[0,r]} \leqslant (n+1)!\, M(s)/(s-r)^{n+1}, \qquad s > r > 0, \tag{1.2}$$

where $M(\cdot)$ is the maximum modulus function. Combining (1.1) and (1.2)

$$\|f - p_n\|_{[0,r]} \leqslant M(s)(r/(s-r))^{n+1}, \qquad s > r. \tag{1.3}$$

THEOREM 1.1. *Let $f(z)$ be an entire function of order ρ, type τ, positive on $[0, \infty)$ and satisfying*

$$\liminf_{r\to\infty} r^{-\rho} \log f(r) = \omega \qquad (0 < \rho < \infty, 0 < \omega \leqslant \tau < \infty).$$

Choose $r = r(n) = \alpha n^{1/\rho}$, and assume that, for all sufficiently large n,

$$f^{(j)}(r(n)) \geqslant 0, \qquad j = 1,..., n,$$

where $\alpha > 0$ is the unique α, minimizing over α, $\beta > 0$ the maximum of

$$-\omega\alpha^\rho \tag{1.4}$$

and

$$\log \alpha - \log \beta + \tau(\alpha + \beta)^\rho. \tag{1.5}$$

Then

$$\limsup_{n\to\infty} \left(\lambda_{0,n}\left(\frac{1}{f}\right)\right)^{n^{-1}} \leqslant \exp(-\omega\alpha^\rho). \tag{1.6}$$

Proof. We first discuss the nonlinear program contained in the statement of the theorem. Let

$$\theta(\alpha, \beta) = \max(-\omega\alpha^\rho, \log \alpha - \log \beta + \tau(\alpha + \beta)^\rho).$$

Since for each $\beta > 0$, (1.4) decreases from 0 to $-\infty$, and (1.5) increases from $-\infty$ to $+\infty$, as α increases from 0, there is a unique $\alpha(\beta)$ minimizing $\theta(\alpha, \beta)$ for each fixed β, and

$$\theta(\alpha(\beta), \beta) = -\omega\alpha(\beta)^\rho = \log \alpha(\beta) - \log \beta + \tau(\alpha(\beta) + \beta)^\rho. \quad (1.7)$$

Choosing $\beta = 1$ and letting $\alpha \to 0^+$ it is clear $\theta(\alpha(1), 1) < 0$ and $\alpha(1) > 0$. Now (1.4) implies

$$\theta(\alpha, \beta) > -\omega\alpha(1)^\rho, \qquad \alpha < \alpha(1),$$

and also

$$\theta(\alpha, \beta) > 0, \qquad \beta \leqslant \alpha.$$

Therefore in seeking to minimize $\theta(\alpha, \beta)$ we may assume $\alpha, \beta \geqslant \alpha(1)$. Given this, it follows from (1.5) that α, β may also be restricted from above. The existence of some minimizing α, β follows from the continuity of (1.4), (1.5), and therefore their maximum, on the compact restricted domain. The unicity of the minimizing α follows from (1.7).

Let $s = s(n) = (\alpha + \beta)n^{1/\rho}$, where α, β are some minimizing pair. Now using the estimate (1.3)

$$\begin{aligned}\| f - p_n \|_{[0,r(n)]} &\leqslant M(s)(r/(s-r))^{n+1} \\ &\leqslant \exp((\tau + \epsilon)\, n(\alpha + \beta)^\rho) \cdot (\alpha/\beta)^{n+1}\end{aligned}$$

where $\epsilon \to 0$ as $n \to \infty$; implying

$$\log(\| f - p_n \|_{[0,r(n)]}) \leqslant O(1) + n[(\tau + \epsilon)(\alpha + \beta)^\rho + \log \alpha - \log \beta]$$

and by (1.7)

$$\limsup \| f - p_n \|_{[0,r(n)]}^{n^{-1}} \leqslant \exp(-\omega\alpha^\rho).$$

Writing

$$\left\| \frac{1}{f} - \frac{1}{p_n} \right\|_{[0,r(n)]} \leqslant \frac{\| f - p_n \|_{[0,r(n)]}}{\inf_{x\in[0,r(n)]} f(x)\, p_n(x)},$$

we conclude

$$\limsup_{n\to\infty} \left\| \frac{1}{f} - \frac{1}{p_n} \right\|_{[0,r(n)]}^{n^{-1}} \leqslant \exp(-\omega\alpha^\rho). \quad (1.8)$$

Note that p_n increases to the right of $r(n)$, for all sufficiently large n, and $p_n(r(n)) = f(r(n))$. Since also

$$f(x) \geqslant \exp((\omega - \epsilon)\, r(n)^\rho), \qquad x \geqslant r(n),$$

where $\epsilon \to 0$ as $n \to \infty$;

$$\left\| \frac{1}{f} - \frac{1}{p_n} \right\|_{[r(n),\infty)} \leqslant 2 \exp(-(\omega - \epsilon)\, \alpha^\rho n), \tag{1.9}$$

where $\epsilon \to 0$ as $n \to \infty$. Combining estimates (1.8) and (1.9) gives the theorem.

Remark 1.2. Certain limits on the degree of relaxation of the positivity conditions are inherent in this argument. In the preceding theorem the positivity conditions (i.e., α) have been chosen so as to give the best result, in terms of order of approximation, using the method of Taylor expansion about the upper end point. Alternatively α_1 (and thus $r(n)$ where $r(n) = \alpha_1 n^{1/\rho}$ and $f^{(j)}(r(n)) \geqslant 0$, $j = 1,..., n$, for all sufficiently large n) could be fixed and the result optimized for this value of α_1. If $\alpha_1 \leqslant \alpha$ and $f^{(j)}(x) \geqslant 0, j = 1,..., n$ $x \geqslant r(n)$, for all sufficiently large n, this "optimal" result will be the previous theorem. If $\alpha_1 > \alpha$ then geometric convergence can still be proved provided

$$\inf_{\beta>0} \log \alpha_1 - \log \beta + \tau(\alpha_1 + \beta)^\rho \tag{1.10}$$

is negative. It is clear that there exists an $\alpha_2 > \alpha$ such that

$$\inf_{\beta>0} \log \alpha_2 - \log \beta + \tau(\alpha_2 + \beta)^\rho = 0,$$

and geometric convergence can be shown with this argument only if $0 \leqslant \alpha_1 < \alpha_2$.

Note very slight modifications of the proof of Theorem 1.1 give a result for Taylor expansion about $r(n)/2$.

THEOREM 1.3. *Let $f(z)$ be an entire function of logarithmic order $\Lambda + 1$, type τ, positive on $[0, \infty)$ and satisfying*

$$\liminf_{r\to\infty} \frac{\log f(r)}{(\log r)^{\Lambda+1}} = \omega, \qquad 0 < \Lambda < \infty, \quad 0 < \omega \leqslant \tau < \infty,$$

Choose $r = r(n) = \exp(\alpha n^{1/\Lambda})$ and assume that for all sufficiently large n

$$f^{(j)}(r(n)) \geqslant 0, \qquad j = 1,..., n,$$

where $\alpha > 0$ is the unique positive solution of the system

$$(\alpha + \beta)^\Lambda = 1/(\tau(1 + \Lambda)), \tag{1.11}$$

$$f_1(\alpha) = f_2(\alpha, \beta), \tag{1.12}$$

where

$$f_1(\alpha) = -\omega\alpha^{1+\Delta}, \tag{1.13}$$

$$f_2(\alpha, \beta) = \tau(\alpha + \beta)^{1+\Delta} - \beta. \tag{1.14}$$

Then

$$\limsup_{n\to\infty} \left(\lambda_{0,n}\left(\frac{1}{f}\right)\right)^{n^{-(1+1/\Delta)}} \leqslant \exp(-\omega\alpha^{1+\Delta}).$$

Proof. First we discuss the nonlinear program:

$$\underset{\alpha,\beta\geqslant 0}{\text{minimize}} \max(f_1(\alpha), f_2(\alpha, \beta)).$$

Let

$$\theta(\alpha, \beta) = \max(-\omega\alpha^{1+\Delta}, \tau(\alpha + \beta)^{1+\Delta} - \beta).$$

Choosing $0 < {\alpha_1}^\Delta < 1/(3^{\Delta+1}\tau)$ we find

$$\theta(\alpha_1, 2\alpha_1) < 0. \tag{1.15}$$

Taking $\alpha^\Delta \geqslant 1/(\tau(1 + \Delta))$ observe (1.14) has a positive minimum as a function of $\beta \geqslant 0$, where $\beta = 0$. Taking $\beta^\Delta \geqslant 1/\tau$ observe (1.14) is nonnegative for all nonnegative α. Thus (1.15) in seeking to minimize $\theta(\alpha, \beta)$ we may assume

$$0 \leqslant \alpha^\Delta \leqslant 1/(\tau(1 + \Delta)), \qquad 0 \leqslant \beta^\Delta \leqslant 1/\tau.$$

The existence of some minimizing α, β now follows from the continuity of (1.13), (1.14) on the compact restricted domain. Write the program equivalently as

$$\underset{0\leqslant\alpha^\Delta\leqslant 1/(\tau(1+\Delta))}{\text{minimize}} \max[f_1(\alpha), \min_{0\leqslant\beta^\Delta\leqslant 1/\tau} f_2(\alpha, \beta)],$$

or

$$\underset{0\leqslant\alpha^\Delta\leqslant 1/(\tau(1+\Delta))}{\text{minimize}} \max[f_1(\alpha), f_2(\alpha, \beta)],$$

where

$$(\alpha + \beta)^\Delta = 1/(\tau(1 + \Delta)).$$

Equation (1.15) and elementary arguments about increasing and decreasing functions now show there is a unique $\alpha, \beta > 0$ solving the program, given by the system in the statement of the theorem.

Using these unique values $\alpha, \beta > 0$, let $s = s(n) = \exp((\alpha + \beta)\, n^{1/A})$. Using the estimate (1.3)

$$\|f - p_n\|_{[0,r(n)]} \leqslant M(s)(r/(s-r))^{n+1},$$

implying

$$\log \|f - p_n\|_{[0,r(n)]} \leqslant O(n^{1/A}) + n^{1+1/A}((\tau + \epsilon)(\alpha + \beta)^{1+A} - \beta), \tag{1.16}$$

where $\epsilon \to 0$ as $n \to \infty$. Since $p_n(x) \geqslant f(r(n))$, $x \geqslant r(n)$, for all large n, and

$$f(x) \geqslant \exp((\omega - \epsilon)\, \alpha^{1+A} n^{1+1/A}), \qquad x \geqslant r(n),$$

where $\epsilon \to 0$ as $n \to \infty$,

$$\left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{[r(n),\infty)} \leqslant 2 \exp(-(\omega - \epsilon)\, \alpha^{1+A} n^{1+1A}), \tag{1.17}$$

where $\epsilon \to 0$ as $n \to \infty$.

Combine (1.16) and (1.17) as in the last theorem to give the estimate of this theorem.

Remark 1.4. Remarks analogous to Remark 1.2 apply to Theorem 1.3.

2. The Degree of Approximation of the Reciprocals of Functions Possessing a Finite Number of Derivatives

In contrast to the case when f is entire relatively few estimates are known for $\lambda_{0,n}(f^{-1})$ when f has only a finite number of derivatives. (There are some results in Blatt [2] and in Freud and Szabados [3].)

For $f \in C^k[-1, 1]$ define

$$E_n^*(f) = \inf_{\{p_n \in \pi_n :\, p_n'(x) \geqslant 0, \forall x \geqslant 1\}} \|f - p_n\|_{[-1,1]} \,.$$

The following estimate of $E_n^*(f)$ is shown in the preprint Beatson [1, Lemma 1 and Corollary 1].

Theorem 2.1. *For each $k = 0, 1, 2,\ldots$, there is a constant D_k such that for each $f \in C^k[-1, 1]$ and $n > k$,*

$$E_n^*(f) \leqslant D_k n^{-k} \omega(f^{(k)}, n^{-1}) + \delta_n(f),$$

where $\omega_{[-1,1]}(f^{(k)}, \cdot)$ *is the modulus of continuity of* $f^{(k)}$ *on* $[-1, 1]$,

$$\begin{aligned} \delta_n(f) &= 0, & k &= 0, \\ &= \max_{j=1,\ldots,k} [\max[0, -f^{(j)}(1)/d_{n,j}]], & k &> 0, \end{aligned}$$

and

$$d_{n,j} = \frac{n^2 \cdot (n^2 - 1) \cdot \quad \cdot (n^2 - (j-1)^2)}{1 \cdot \quad 3 \quad \cdot \quad \cdot (2j - 1)}.$$

Estimates of $\lambda_{0,n}(f^{-1})$ will be derived from the following corollary to Theorem 2.1.

THEOREM 2.2. *For each* $k = 0, 1, 2,\ldots,$ *there is a constant* D_k *such that for each* $r > 0, f \in C^k[0, r]$, *and* $n > k$, *there exists a polynomial* $p_n \in \pi_n$ *with*

$$p_n'(x) \geqslant 0, \qquad \forall x \geqslant r;$$

and

$$\|f - p_n\|_{[0,r]} \leqslant D_k n^{-k} r^k \omega_{[0,r]}(f^{(k)}, n^{-1}r) + \delta_n(f, r),$$

where $\omega_{[0,r]}(f^{(k)}, \cdot)$ *is the modulus of continuity of* $f^{(k)}$ *on* $[0, r]$,

$$\begin{aligned} \delta_n(f, r) &= 0, & k &= 0, \\ &= \max_{j=1,\ldots,k} \max(0, -[f^{(j)}(r)(r/2)^j]/d_{n,j}), & k &> 0; \end{aligned}$$

and

$$d_{n,j} = \frac{n^2 \cdot (n^2 - 1) \cdot \quad \cdot (n^2 - (j-1)^2)}{1 \cdot \quad 3 \quad \cdot \quad \cdot (2j - 1)}.$$

Proof. Consider the transformation of $f(x) \in C^k[0, r]$ into $g(y) \in C^k[-1, 1]$ given by

$$y = -1 + (2/r)x \qquad \text{and} \qquad g(y) = f(x). \tag{2.1}$$

This transformation is invertible, takes polynomials into polynomials of the same degree, and transforms derivatives according to

$$\frac{d^j g}{dy^j}(y) = \left(\frac{r}{2}\right)^j \frac{d^j f}{dx^j}(x). \tag{2.2}$$

Hence, in particular,

$$\omega_{[-1,1]}(g^{(k)}, n^{-1}) = (r/2)^k\, \omega_{[0,r]}(f^{(k)}, n^{-1}(r/2)) \leqslant r^k \omega_{[0,r]}(f^{(k)}, n^{-1}r). \tag{2.3}$$

Fix k, $n > k$, and r. An application of Theorem 2.1 together with (2.2) and (2.3) shows the existence of a polynomial $h_n \in \pi_n$ with

$$h_n'(y) \geqslant 0, \qquad \forall y \geqslant 1,$$

and

$$\| g - h_n \|_{[-1,1]} \leqslant D_k n^{-k} r^k \omega_{[0,r]}(f^{(k)}, n^{-1}r) + \delta_n(f, r).$$

Hence inverting the transformation (2.1), there exists a polynomial $p_n \in \pi_n$ with

$$p_n'(x) \geqslant 0, \qquad \forall x \geqslant r,$$

and

$$\| f - p_n \|_{[0,r]} \leqslant D_k n^{-k} r^k \omega_{[0,r]}(f^{(k)}, n^{-1}r) + \delta_n(f, r). \quad \blacksquare$$

Clearly Theorem 2.2 has many corollaries concerning rational approximation on $[0, \infty)$. We cite three typical examples.

COROLLARY 2.3. *Suppose for some positive integer k the function $f \in C^k[0, \infty)$ satisfies*

$$\| f^{(k)} \|_{[0,r]} \leqslant p(r)\, g(r), \qquad f(x) \geqslant g(r), \qquad \forall x \geqslant r \geqslant 0, \tag{2.4}$$

where $g(r)$ is a positive increasing function in $C[0, \infty)$, $p(r)$ is a positive function with

$$\lim_{r\to\infty} \frac{\max(\log p(r), \log r)}{\log g(r)} = 0, \tag{2.5}$$

and

$$[1 - \operatorname{sign}(f^{(j)}(r))] f^{(j)}(r) \leqslant p(r)[g(r)]^{(4j-k)/k}, \qquad j = 1,\ldots, k, \tag{2.6}$$

for all sufficiently large r. Then

$$\lambda_{0,n}\left(\frac{1}{f}\right) = O(n^{(-k/2)+\epsilon}), \qquad \forall \epsilon > 0.$$

Proof. Take $N_1 > k$ so large that for $n > N_1$ there exists $r = r(n) > 0$ such that

$$g(r(n)) = n^{k/2}. \tag{2.7}$$

Take $N_2 \geqslant N_1$ so large that (2.6) applies whenever $r \geqslant r(N_2)$. In the following assume $n \geqslant N_2$. Note (2.5) and (2.7) imply

$$r(n)^k = O(n^\epsilon) \quad \text{and} \quad p(r(n)) = O(n^\epsilon), \qquad \forall \epsilon > 0. \tag{2.8}$$

Apply Theorem 2.2 to f on $[0, r(n)]$. By (2.4)

$$\omega_{[0,r]}(f^k, n^{-1}r) \leqslant 2\,\|f^{(k)}\|_{[0,r]} \leqslant 2p(r)\,g(r);$$

which together with (2.7) and (2.8) implies

$$D_k n^{-k} r^k \omega_{[0,r]}(f^{(k)}, n^{-1}r) \leqslant n^{-k}\cdot O(n^{\epsilon})\cdot n^{k/2} \leqslant O(n^{(-k/2)+\epsilon}), \qquad \forall\epsilon > 0.$$

Equations (2.6), (2.7), and (2.8) imply

$$\begin{aligned}\delta_n(f, r) &\leqslant \max_{j=1,\ldots,k} p(r)[g(r)]^{(4j-k)/k}\cdot (r/2)^j/d_{n,j}\\ &= O(n^{\epsilon}) \max_{j=1,\ldots,k} n^{2j-(k/2)}\cdot n^{-2j} = O(n^{(-k/2)+\epsilon}), \qquad \forall\epsilon > 0.\end{aligned}$$

Hence Theorem 2.2 guarantees the existence of a sequence of polynomials $\{p_n \in \pi_n\}_{n=N_2}^{\infty}$ with

$$p_n'(x) \geqslant 0, \ \ \forall x \geqslant r(n), \qquad \text{and} \qquad \|f - p_n\|_{[0,r(n)]} = O(n^{(-k/2)+\epsilon}), \ \ \forall\epsilon > 0. \tag{2.9}$$

By (2.9) p_n is positive on $[0, \infty)$ for all sufficiently large n, and

$$\left\|\frac{1}{f} - \frac{1}{p_n}\right\|_{[0,r(n)]} \leqslant \frac{\|f - p_n\|_{[0,r(n)]}}{\inf_{x\in[0,r(n)]} f(x)\,p_n(x)} = O(n^{(-k/2)+\epsilon}). \tag{2.10}$$

For the interval $[r(n), \infty)$, (2.4) and (2.9) imply $f(x) \geqslant g(r(n))$ and $p_n(x) \geqslant p_n(r(n)) \geqslant g(r(n)) + o(1)$, $\forall x \geqslant r(n)$. Hence by (2.7), provided n is so large that $p_n(r(n)) > 0$,

$$\left\|\frac{1}{f(x)} - \frac{1}{p_n(x)}\right\|_{[r(n),\infty)} \leqslant \left\|\frac{1}{f(x)}\right\|_{[r(n),\infty)} + \left\|\frac{1}{p_n(x)}\right\|_{[r(n),\infty)} = O(n^{-k/2}). \tag{2.11}$$

Equations (2.10) and (2.11) together imply the corollary. ∎

COROLLARY 2.4. *There is a constant C such that for each function $f_\alpha = (1 + x)^\alpha$, $0 < \alpha < 1$;*

$$\lambda_{0,n}\left(\frac{1}{f_\alpha}\right) \leqslant Cn^{-\alpha/(1+\alpha)}, \qquad n \geqslant 1.$$

Proof. Let $\beta = 1/(1 + \alpha)$. According to Theorem 2.2 there exists a sequence of polynomials $\{p_n \in \pi_n\}_{n=1}^{\infty}$ with

$$\|f_\alpha - p_n\|_{[0,n^\beta]} \leqslant D_0\omega_{[0,n^\beta]}(f, n^{\beta-1}), \qquad \text{and} \qquad p_n'(x) \geqslant 0, \qquad x \geqslant n^\beta.$$

Let

$$q_n(x) = p_n(x) + D_0\omega_{[0,n^\beta]}(f_\alpha, n^{\beta-1}) = p_n(x) + D_0[(1 + n^{\beta-1})^\alpha - 1]$$

then

$$q_n(x) \geqslant f_\alpha(x) \geqslant 1, \qquad x \in [0, n^\beta], \tag{2.12}$$

$$\|f_\alpha(x) - q_n(x)\|_{[0,n^\beta]} \leqslant 2D_0 n^{\beta-1}, \tag{2.13}$$

and

$$f_\alpha(x) \geqslant n^{\alpha\beta}, \qquad \forall x \geqslant n^\beta. \tag{2.14}$$

Equations (2.12), (2.13), (2.14), and an argument analogous to the latter part of the last corollary show

$$\left\| \frac{1}{f_\alpha} - \frac{1}{q_n} \right\|_{[0,\infty)} \leqslant n^{-\alpha/(1+\alpha)} \cdot \max(2D_0, 2), \qquad n = 1, 2, \ldots. \quad ∎$$

COROLLARY 2.5. *Suppose the function $f \in C^\infty[0, \infty)$ satisfies the following conditions*:

(1) *To each derivative $f^{(k)}$ there corresponds a real number R_k such that $f^{(k)}(x) \geqslant 0$ for all $x \geqslant R_k$.*

(2) *There exist real numbers $R_0 > 0$, $\beta > 0$, and a positive increasing function $g \in C[0, \infty)$ with*

$$\lim_{r\to\infty} [\log r/\log g(r)] = 0; \tag{2.15}$$

such that

$$f(x) \geqslant g(r), \qquad \forall x \geqslant r \geqslant 0, \tag{2.16}$$

and

$$\|f^{(k)}\|_{[0,r]} \leqslant g(r)^\beta, \qquad \forall k \geqslant 1 \text{ and } \forall r \geqslant R_0. \tag{2.17}$$

Then

$$\lambda_{0,n}\left(\frac{1}{f}\right) = O(n^{-j}), \qquad j = 1, 2, 3, \ldots.$$

Proof. Let k be a fixed, but arbitrary, positive integer. By (2.15) for all sufficiently large n there exists $r(n) > 0$ such that $g(r(n)) = n^{k/(2\beta)}$. Using (2.15) again, $r(n) = O(n^\epsilon)$, $\forall \epsilon > 0$. By hypothesis (1), $f^{(j)}(r(n)) \geqslant 0$, $j = 1, \ldots, k$, for all sufficiently large n. Using (2.17) and applying Theorem 2.2

(in the manner of the initial part of Corollary 2.3) there exists a sequence of polynomials $\{p_n\}$ such that

$$\|f - p_n\|_{[0,r(n)]} = O(n^{(-k/2)+\epsilon}), \qquad \forall \epsilon > 0,$$
$$p_n'(x) \geqslant 0, \qquad \forall x \geqslant r(n),$$

and also

$$f(x) \geqslant g(r(n)) = n^{k/(2\beta)}, \qquad \forall x \geqslant r(n).$$

It follows, by the standard argument, that p_n is positive on $[0, \infty)$ for all sufficiently large n, and

$$\lambda_{0,n}(1/f) \leqslant \left\| \frac{1}{f} - \frac{1}{p_n} \right\|_{[0,\infty)} = O(n^{(-k/2)+\epsilon}) + O(n^{-k/2\beta}), \qquad \forall \epsilon > 0.$$

Since k was an arbitrary positive integer this implies

$$\lambda_{0,n}(1/f) = O(n^{-j}), \qquad \forall j = 1, 2, 3,\dots . \quad \blacksquare$$

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